

# High frequency dispersive estimates in dimension two

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## Abstract

We prove dispersive estimates at high frequency in dimension two for both the wave and the Schrödinger groups for a very large class of real-valued potentials.

## 1 Introduction and statement of results

The purpose of this note is to prove dispersive estimates at high frequency for the wave group  $e^{it\sqrt{G}}$  and the Schrödinger group  $e^{itG}$ , where  $G$  denotes the self-adjoint realization of the operator  $-\Delta + V$  on  $L^2(\mathbf{R}^2)$  and  $V$  is a real-valued potential which decays at infinity in a way that  $G$  has no real resonances nor eigenvalues in an interval  $[a_0, +\infty)$ ,  $a_0 > 0$ . In fact, we are looking for as large as possible class of potentials for which we have dispersive estimates similar to those we do for the free operator  $G_0$ . Hereafter  $G_0$  denotes the self-adjoint realization of the operator  $-\Delta$  on  $L^2(\mathbf{R}^2)$ . It turns out that in dimension two one can get such dispersive estimates at high frequency for potentials satisfying

$$\sup_{y \in \mathbf{R}^2} \int_{\mathbf{R}^2} \frac{|V(x)|dx}{|x-y|^{1/2}} \leq C < +\infty. \quad (1.1)$$

Clearly, (1.1) is fulfilled for potentials  $V \in L^\infty(\mathbf{R}^2)$  satisfying

$$|V(x)| \leq C\langle x \rangle^{-\delta}, \quad \forall x \in \mathbf{R}^2, \quad (1.2)$$

with constants  $C > 0$ ,  $\delta > 3/2$ . Given any  $a > 0$ , set  $\chi_a(\sigma) = \chi_1(\sigma/a)$ , where  $\chi_1 \in C^\infty(\mathbf{R})$ ,  $\chi_1(\sigma) = 0$  for  $\sigma \leq 1$ ,  $\chi_1(\sigma) = 1$  for  $\sigma \geq 2$ . Our first result is the following

**Theorem 1.1** *Let  $V$  satisfy (1.1). Then, there exists a constant  $a_0 > 0$  so that for every  $a \geq a_0$ ,  $0 < \epsilon \ll 1$ ,  $2 \leq p < +\infty$ , we have the estimates*

$$\left\| e^{it\sqrt{G}} G^{-3/4-\epsilon} \chi_a(G) \right\|_{L^1 \rightarrow L^\infty} \leq C_\epsilon |t|^{-1/2}, \quad t \neq 0, \quad (1.3)$$

$$\left\| e^{it\sqrt{G}} G^{-3\alpha/4} \chi_a(G) \right\|_{L^{p'} \rightarrow L^p} \leq C |t|^{-\alpha/2}, \quad t \neq 0, \quad (1.4)$$

where  $1/p + 1/p' = 1$ ,  $\alpha = 1 - 2/p$ .

The estimate (1.3) is proved in [2] under the assumption (1.2). Moreover, if in addition one supposes that  $G$  has no strictly positive resonances, it is shown in [2] that (1.3) holds for any  $a > 0$  still under (1.2). In dimension three an analogue of (1.3) is proved in [2], [4] for potentials satisfying (1.2) with  $\delta > 2$ , and extended in [3] to a large subset of potentials satisfying

$$\sup_{y \in \mathbf{R}^3} \int_{\mathbf{R}^3} \frac{|V(x)|dx}{|x-y|} \leq C < +\infty. \quad (1.5)$$

In dimensions  $n \geq 4$  there are very few results. In [1], an analogue of (1.3) is proved for potentials belonging to the Schwartz class, while in [11] dispersive estimates with a loss of  $(n-3)/2$  derivatives are obtained for potentials satisfying (1.2) with  $\delta > (n+1)/2$ . Recently, in [7] dispersive estimates at low frequency have been proved in dimensions  $n \geq 4$  for a very large class of potentials, provided zero is neither an eigenvalue nor a resonance.

Our second result is the following

**Theorem 1.2** *Let  $V$  satisfy (1.1). Then, there exists a constant  $a_0 > 0$  so that for every  $a \geq a_0$ , we have the estimate*

$$\|e^{itG}\chi_a(G)\|_{L^1 \rightarrow L^\infty} \leq C|t|^{-1}, \quad t \neq 0. \quad (1.6)$$

Note that the estimate (1.6) (for any  $a > 0$ ) is proved in [9] for potentials satisfying (1.2) with  $\delta > 2$ . In dimension three an analogue of (1.6) (for any  $a > 0$ ) is proved in [10] for potentials satisfying (1.5) with  $C > 0$  small enough, and in [5] for potentials  $V \in L^{3/2-\epsilon} \cap L^{3/2+\epsilon}$ ,  $0 < \epsilon \ll 1$ , not necessarily small. In dimensions  $n \geq 4$ , an analogue of (1.6) (for any  $a > 0$ ) is proved in [6] for potentials satisfying (1.2) with  $\delta > n$  as well as the condition  $\widehat{V} \in L^1$ . This result has been recently extended in [8] to potentials satisfying (1.2) with  $\delta > n-1$  and  $\widehat{V} \in L^1$ . Note also the work [12], where an analogue of (1.6) (for any  $a > 0$ ) with a loss of  $(n-3)/2$  derivatives is obtained for potentials satisfying (1.2) with  $\delta > (n+2)/2$ . In [8] dispersive estimates at low frequency have been also proved in dimensions  $n \geq 4$  for a very large class of potentials, provided zero is neither an eigenvalue nor a resonance.

To prove (1.3) we use the same idea we have already used in [7] to prove low frequency dispersive estimates in dimensions  $n \geq 4$ . The key point is the following estimate which holds in all dimensions  $n \geq 2$ :

$$h \int_{-\infty}^{\infty} \left\| V e^{it\sqrt{G_0}} \psi(h^2 G_0) f \right\|_{L^1} dt \leq \gamma_n C_n(V) h^{-(n-3)/2} \|f\|_{L^1}, \quad h > 0, \quad (1.7)$$

where  $\psi \in C_0^\infty((0, +\infty))$ ,  $\gamma_n > 0$  is a constant independent of  $V$ ,  $h$  and  $f$ , and

$$C_n(V) := \sup_{y \in \mathbf{R}^n} \int_{\mathbf{R}^n} \frac{|V(x)| dx}{|x-y|^{(n-1)/2}} < +\infty. \quad (1.8)$$

Our approach is based on the observation that if

$$C_n(V) h^{-(n-3)/2} \ll 1, \quad (1.9)$$

then (1.7) implies (under reasonable assumptions on the potential) a similar estimate for the perturbed wave group, namely

$$h \int_{-\infty}^{\infty} \left\| V e^{it\sqrt{G}} \psi(h^2 G) f \right\|_{L^1} dt \leq \widetilde{C}_n(V) h^{-(n-3)/2} \|f\|_{L^1}. \quad (1.10)$$

When  $n = 3$ , (1.9) is fulfilled for small potentials and all  $h$ , when  $n \geq 4$ , (1.9) is fulfilled for large  $h$  (i.e. at low frequency) without extra restrictions on the potential, while for  $n = 2$ , (1.9) is fulfilled for small  $h$  (i.e. at high frequency) again without restrictions on the potential others than (1.1). Note that (1.10) may hold without (1.9). Indeed, when  $n = 3$ , (1.10) is proved in [5] for potentials  $V \in L^{3/2-\epsilon} \cap L^{3/2+\epsilon}$  and all  $h > 0$ , and then used to prove the three dimensional analogue of (1.6). In the present paper we adapt this approach to the case of dimension two, and show that (1.6) follows from (1.10) for potentials satisfying (1.1) only, provided the parameter  $a$  is taken large enough (see Section 3).

## 2 Proof of Theorem 1.1

Let  $\psi \in C_0^\infty((0, +\infty))$  and set

$$\Phi(t; h) = e^{it\sqrt{G}}\psi(h^2G) - e^{it\sqrt{G_0}}\psi(h^2G_0).$$

We will first show that (1.3) and (1.4) follow from the following

**Proposition 2.1** *Let  $V$  satisfy (1.1). Then, there exist positive constants  $C$  and  $h_0$  so that for  $0 < h \leq h_0$  we have*

$$\|\Phi(t; h)\|_{L^1 \rightarrow L^\infty} \leq Ch^{-1}|t|^{-1/2}, \quad t \neq 0. \quad (2.1)$$

Writing

$$\sigma^{-3/4-\epsilon}\chi_a(\sigma) = \int_0^{a^{-1}} \psi(\sigma\theta)\theta^{-1/4+\epsilon}d\theta,$$

where  $\psi(\sigma) = \sigma^{1/4-\epsilon}\chi_1'(\sigma) \in C_0^\infty((0, +\infty))$ , and using (2.1) we get

$$\begin{aligned} & \left\| e^{it\sqrt{G}}G^{-3/4-\epsilon}\chi_a(G) - e^{it\sqrt{G_0}}G_0^{-3/4-\epsilon}\chi_a(G_0) \right\|_{L^1 \rightarrow L^\infty} \\ & \leq \int_0^{a^{-1}} \left\| \Phi(t; \sqrt{\theta}) \right\|_{L^1 \rightarrow L^\infty} \theta^{-1/4+\epsilon}d\theta \leq C|t|^{-1/2} \int_0^{a^{-1}} \theta^{-3/4+\epsilon}d\theta \leq C|t|^{-1/2}, \end{aligned} \quad (2.2)$$

provided  $a$  is taken large enough. Clearly, (1.3) follows from (2.2) and the fact that it holds for  $G_0$ . To prove (1.4), observe that an interpolation between (2.1) and the trivial bound

$$\|\Phi(t; h)\|_{L^2 \rightarrow L^2} \leq C$$

yields

$$\|\Phi(t; h)\|_{L^{p'} \rightarrow L^p} \leq Ch^{-\alpha}|t|^{-\alpha/2}, \quad t \neq 0, \quad (2.3)$$

for every  $2 \leq p \leq +\infty$ ,  $p'$  and  $\alpha$  being as in Theorem 1.1. Now we write

$$\sigma^{-3\alpha/4}\chi_a(\sigma) = \int_0^{a^{-1}} \psi(\sigma\theta)\theta^{-1+3\alpha/4}d\theta,$$

and use (2.3) to obtain (for  $0 < \alpha \leq 1$ )

$$\begin{aligned} & \left\| e^{it\sqrt{G}}G^{-3\alpha/4}\chi_a(G) - e^{it\sqrt{G_0}}G_0^{-3\alpha/4}\chi_a(G_0) \right\|_{L^{p'} \rightarrow L^p} \\ & \leq \int_0^{a^{-1}} \left\| \Phi(t; \sqrt{\theta}) \right\|_{L^{p'} \rightarrow L^p} \theta^{-1+3\alpha/4}d\theta \leq C|t|^{-\alpha/2} \int_0^{a^{-1}} \theta^{-1+\alpha/4}d\theta \leq C|t|^{-\alpha/2}, \end{aligned} \quad (2.4)$$

provided  $a$  is taken large enough. Now, (1.4) follows from (2.4) and the fact that it holds for  $G_0$ .

*Proof of Proposition 2.1.* We will first prove the following

**Lemma 2.2** *Let  $V$  satisfy (1.1). Then, there exist positive constants  $C$  and  $h_0$  so that for  $0 < h \leq h_0$  we have*

$$\|\psi(h^2G) - \psi(h^2G_0)\|_{L^1 \rightarrow L^1} \leq Ch^{1/2}. \quad (2.5)$$

*Proof.* We will make use of the formula

$$\psi(h^2 G) = \frac{2}{\pi} \int_{\mathbf{C}} \frac{\partial \tilde{\varphi}}{\partial \bar{z}}(z) (h^2 G - z^2)^{-1} z L(dz), \quad (2.6)$$

where  $L(dz)$  denotes the Lebesgue measure on  $\mathbf{C}$ ,  $\tilde{\varphi} \in C_0^\infty(\mathbf{C})$  is an almost analytic continuation of  $\varphi(\lambda) = \psi(\lambda^2)$  supported in a small complex neighbourhood of  $\text{supp } \varphi$  and satisfying

$$\left| \frac{\partial \tilde{\varphi}}{\partial \bar{z}}(z) \right| \leq C_N |\text{Im } z|^N, \quad \forall N \geq 1.$$

For  $\pm \text{Im } \lambda \geq 0$ ,  $\text{Re } \lambda > 0$ , set

$$R_0^\pm(\lambda) = (G_0 - \lambda^2)^{-1}, \quad R^\pm(\lambda) = (G - \lambda^2)^{-1}.$$

We have the identity

$$R^\pm(\lambda) (1 + V R_0^\pm(\lambda)) = R_0^\pm(\lambda). \quad (2.7)$$

It is well known that the kernels of the operators  $R_0^\pm(\lambda)$  are given in terms of the zero order Hankel functions by the formula

$$[R_0^\pm(\lambda)](x, y) = \pm i 4^{-1} H_0^\pm(\lambda |x - y|).$$

Moreover, the functions  $H_0^\pm$  satisfy the bound

$$|H_0^\pm(\lambda)| \leq C |\lambda|^{-1/2} e^{-|\text{Im } \lambda|}, \quad |\lambda| \geq 1, \pm \text{Im } \lambda \geq 0, \quad (2.8)$$

while near  $\lambda = 0$  they are of the form

$$H_0^\pm(\lambda) = a_{0,1}^\pm(\lambda) + a_{0,2}^\pm(\lambda) \log \lambda, \quad (2.9)$$

where  $a_{0,j}^\pm$  are analytic functions. In particular, we have

$$|H_0^\pm(\lambda)| \leq C |\lambda|^{-1/2}, \quad \text{Re } \lambda > 0, \pm \text{Im } \lambda \geq 0. \quad (2.10)$$

Using these bounds we will prove the following

**Lemma 2.3** *Let  $V$  satisfy (1.1). Then, there exist constants  $C > 0$  and  $0 < h_0 \leq 1$  so that for  $z \in \mathbf{C}_\varphi^\pm := \{z \in \text{supp } \tilde{\varphi}, \pm \text{Im } z \geq 0\}$ , we have the estimates*

$$\|V R_0^\pm(z/h)\|_{L^1 \rightarrow L^1} \leq C h^{1/2}, \quad 0 < h \leq 1, \quad (2.11)$$

$$\|V R^\pm(z/h)\|_{L^1 \rightarrow L^1} \leq C h^{1/2}, \quad 0 < h \leq h_0, \quad (2.12)$$

$$\|R_0^\pm(z/h)\|_{L^1 \rightarrow L^1} \leq C h^2 |\text{Im } z|^{-2}, \quad 0 < h \leq 1, \text{Im } z \neq 0, \quad (2.13)$$

$$\|R^\pm(z/h)\|_{L^1 \rightarrow L^1} \leq C h^2 |\text{Im } z|^{-2}, \quad 0 < h \leq h_0, \text{Im } z \neq 0. \quad (2.14)$$

*Proof.* By (1.1) and (2.10), the norm in the LHS of (2.11) is upper bounded by

$$\sup_{y \in \mathbf{R}^2} \int_{\mathbf{R}^2} |V(x)| |H_0^\pm(z|x - y|/h)| dx \leq C h^{1/2} \int_{\mathbf{R}^2} \frac{|V(x)| dx}{|x - y|^{1/2}} \leq C' h^{1/2}.$$

Similarly, the norm in the LHS of (2.13) is upper bounded by

$$\sup_{y \in \mathbf{R}^2} \int_{\mathbf{R}^2} |H_0^\pm(z|x - y|/h)| dx = h^2 \sup_{y \in \mathbf{R}^2} \int_{\mathbf{R}^2} |H_0^\pm(z|x - y|)| dx$$

$$\leq Ch^2 |\operatorname{Im} z|^{-2} \int_{\mathbf{R}^2} \langle x-y \rangle^{-3/2} |x-y|^{-1} dx = C' h^2 |\operatorname{Im} z|^{-2} \int_0^\infty \langle \sigma \rangle^{-3/2} d\sigma.$$

To prove (2.12) and (2.14) we will make use of the identity (2.7). It follows from (2.11) that there exists a constant  $0 < h_0 \leq 1$  so that for  $0 < h \leq h_0$  the operator  $1 + VR_0^\pm(z/h)$  is invertible on  $L^1$  with an inverse satisfying

$$\left\| (1 + VR_0^\pm(z/h))^{-1} \right\|_{L^1 \rightarrow L^1} \leq C, \quad z \in \mathbf{C}_\varphi^\pm, \quad (2.15)$$

with a constant  $C > 0$  independent of  $z$  and  $h$ . Clearly, (2.12) follows from (2.11) and (2.15), while (2.14) follows from (2.13) and (2.15).  $\square$

To prove (2.5) we rewrite the identity (2.7) in the form

$$R^\pm(z/h) - R_0^\pm(z/h) = R_0^\pm(z/h) VR_0^\pm(z/h) (1 + VR_0^\pm(z/h))^{-1},$$

and hence, using Lemma 2.3 and (2.15), we get

$$\left\| h^{-2} R^\pm(z/h) - h^{-2} R_0^\pm(z/h) \right\|_{L^1 \rightarrow L^1} \leq Ch^{1/2} |\operatorname{Im} z|^{-2}, \quad 0 < h \leq h_0, z \in \mathbf{C}_\varphi^\pm, \operatorname{Im} z \neq 0. \quad (2.16)$$

It is easy now to see that (2.5) follows from (2.6) and (2.16).  $\square$

We will now derive (2.1) from the following

**Proposition 2.4** *Let  $V$  satisfy (1.1). Then, there exist positive constants  $C$  and  $h_0$  so that we have, for  $0 \leq s \leq 1/2$ ,  $f, g \in L^1$ ,*

$$\left\| e^{it\sqrt{G_0}} \psi(h^2 G_0) f \right\|_{L^\infty} \leq Ch^{-3/2} |t|^{-1/2} \|f\|_{L^1}, \quad h > 0, t \neq 0, \quad (2.17)$$

$$\begin{aligned} \int_{\mathbf{R}^2} \int_{\mathbf{R}^2} \int_{-\infty}^\infty |t|^s |x-y|^{-s} \left| V e^{it\sqrt{G_0}} \psi(h^2 G_0) f(x) \right| |g(y)| dt dx dy \\ \leq Ch^{-1/2} \|f\|_{L^1} \|g\|_{L^1}, \quad h > 0, \end{aligned} \quad (2.18)$$

$$\begin{aligned} \int_{\mathbf{R}^2} \int_{\mathbf{R}^2} \int_{-\infty}^\infty |t|^s \langle |x-y|/h \rangle^{-s} \left| V e^{it\sqrt{G}} \psi(h^2 G) f(x) \right| |g(y)| dt dx dy \\ \leq Ch^{s-1/2} \|f\|_{L^1} \|g\|_{L^1}, \quad 0 < h \leq h_0. \end{aligned} \quad (2.19)$$

As in [12], using Duhamel's formula

$$e^{it\sqrt{G}} = e^{it\sqrt{G_0}} + i \frac{\sin(t\sqrt{G_0})}{\sqrt{G_0}} \left( \sqrt{G} - \sqrt{G_0} \right) - \int_0^t \frac{\sin((t-\tau)\sqrt{G_0})}{\sqrt{G_0}} V e^{i\tau\sqrt{G}} d\tau$$

we get the identity

$$\Phi(t; h) = \sum_{j=1}^2 \Phi_j(t; h), \quad (2.20)$$

where

$$\begin{aligned} \Phi_1(t; h) &= (\psi_1(h^2 G) - \psi_1(h^2 G_0)) e^{it\sqrt{G}} \psi(h^2 G) \\ &\quad + \psi_1(h^2 G_0) e^{it\sqrt{G_0}} (\psi(h^2 G) - \psi(h^2 G_0)) \\ &\quad - i\psi_1(h^2 G_0) \sin(t\sqrt{G_0}) (\psi(h^2 G) - \psi(h^2 G_0)) \\ &\quad + i\tilde{\psi}_1(h^2 G_0) \sin(t\sqrt{G_0}) (\tilde{\psi}(h^2 G) - \tilde{\psi}(h^2 G_0)), \end{aligned}$$

$$\Phi_2(t; h) = -h \int_0^t \tilde{\psi}_1(h^2 G_0) \sin\left((t-\tau)\sqrt{G_0}\right) V e^{i\tau\sqrt{G}} \psi(h^2 G) d\tau,$$

where  $\psi_1 \in C_0^\infty((0, +\infty))$ ,  $\psi_1 = 1$  on  $\text{supp } \psi$ ,  $\tilde{\psi}(\sigma) = \sigma^{1/2} \psi(\sigma)$ ,  $\tilde{\psi}_1(\sigma) = \sigma^{-1/2} \psi_1(\sigma)$ . By Proposition 2.4 and (2.5), we have

$$\begin{aligned} \|\Phi_1(t; h)f\|_{L^\infty} &\leq Ch^{-1}|t|^{-1/2}\|f\|_{L^1} + Ch^{1/2}\|\Phi(t; h)f\|_{L^\infty}, \\ &t^{1/2}|\langle \Phi_2(t; h)f, g \rangle| \\ &\leq h \int_0^{t/2} (t-\tau)^{1/2} \left\| \sin\left((t-\tau)\sqrt{G_0}\right) \tilde{\psi}_1(h^2 G_0)g \right\|_{L^\infty} \left\| V e^{i\tau\sqrt{G}} \psi(h^2 G)f \right\|_{L^1} d\tau \\ &\quad + h \int_{t/2}^t \left\| V \sin\left((t-\tau)\sqrt{G_0}\right) \tilde{\psi}_1(h^2 G_0)g \right\|_{L^1} \tau^{1/2} \left\| e^{i\tau\sqrt{G}} \psi(h^2 G)f \right\|_{L^\infty} d\tau \\ &\leq Ch^{-1/2}\|g\|_{L^1} \int_{-\infty}^{\infty} \left\| V e^{i\tau\sqrt{G}} \psi(h^2 G)f \right\|_{L^1} d\tau \\ &\quad + h \sup_{t/2 \leq \tau \leq t} \tau^{1/2} \left\| e^{i\tau\sqrt{G}} \psi(h^2 G)f \right\|_{L^\infty} \int_{-\infty}^{\infty} \left\| V \sin\left((t-\tau)\sqrt{G_0}\right) \tilde{\psi}_1(h^2 G_0)g \right\|_{L^1} d\tau \\ &\leq Ch^{-1}\|g\|_{L^1}\|f\|_{L^1} + Ch^{1/2}\|g\|_{L^1} \sup_{t/2 \leq \tau \leq t} \tau^{1/2} \left\| e^{i\tau\sqrt{G}} \psi(h^2 G)f \right\|_{L^\infty}, \end{aligned} \tag{2.21}$$

for  $t > 0$ , which clearly implies

$$t^{1/2}\|\Phi_2(t; h)f\|_{L^\infty} \leq Ch^{-1}\|f\|_{L^1} + Ch^{1/2} \sup_{t/2 \leq \tau \leq t} \tau^{1/2} \left\| e^{i\tau\sqrt{G}} \psi(h^2 G)f \right\|_{L^\infty}. \tag{2.22}$$

By (2.20)-(2.22), we conclude

$$\begin{aligned} t^{1/2}\|\Phi(t; h)f\|_{L^\infty} &\leq Ch^{-1}\|f\|_{L^1} + Ch^{1/2}t^{1/2}\|\Phi(t; h)f\|_{L^\infty} \\ &\quad + Ch^{1/2} \sup_{t/2 \leq \tau \leq t} \tau^{1/2} \|\Phi(\tau; h)f\|_{L^\infty}. \end{aligned} \tag{2.23}$$

Taking  $h$  small enough we can absorb the second and the third terms in the RHS of (2.23), thus obtaining (2.1). Clearly, the case of  $t < 0$  can be treated in the same way.  $\square$

*Proof of Proposition 2.3.* The kernel of the operator  $e^{it\sqrt{G_0}}\psi(h^2 G_0)$  is of the form  $K_h(|x-y|, t)$ , where

$$K_h(\sigma, t) = (2\pi)^{-1} \int_0^\infty e^{it\lambda} J_0(\sigma\lambda) \psi(h^2 \lambda^2) \lambda d\lambda = h^{-2} K_1(\sigma h^{-1}, t h^{-1}), \tag{2.24}$$

where  $J_0(z) = (H_0^+(z) + H_0^-(z))/2$  is the Bessel function of order zero. It is shown in [12] (Section 2) that  $K_h$  satisfies the estimates (for all  $\sigma, h > 0, t \neq 0$ )

$$|K_1(\sigma, t)| \leq C|t|^{-s} \langle \sigma \rangle^{s-1/2}, \quad \forall s \geq 0, \tag{2.25}$$

$$|K_h(\sigma, t)| \leq Ch^{-3/2}|t|^{-s} \sigma^{s-1/2}, \quad 0 \leq s \leq 1/2. \tag{2.26}$$

Clearly, (2.17) follows from (2.26) with  $s = 1/2$ . It is easy also to see that (2.18) follows from (1.1) and the following

**Lemma 2.5** For all  $0 \leq s \leq 1/2$ ,  $\sigma, h > 0$ , we have

$$\int_{-\infty}^{\infty} \langle t/h \rangle^s |K_h(\sigma, t)| dt \leq Ch^{-1} \langle \sigma/h \rangle^{s-1/2}, \quad (2.27)$$

$$\int_{-\infty}^{\infty} |t|^s |K_h(\sigma, t)| dt \leq Ch^{-1/2} \sigma^{s-1/2}. \quad (2.28)$$

*Proof.* Clearly, (2.28) follows from (2.27). It is also clear from (2.24) that it suffices to prove (2.27) with  $h = 1$ . When  $0 < \sigma \leq 1$ , this follows from (2.25). Let now  $\sigma \geq 1$ . We decompose  $K_1$  as  $K_1^+ + K_1^-$ , where  $K_1^\pm$  is defined by replacing in (2.24) the function  $J_0$  by  $H_0^\pm/2$ . Recall that  $H_0^\pm(z) = e^{\pm iz} b_0^\pm(z)$ , where  $b_0^\pm(z)$  is a symbol of order  $-1/2$  for  $z \geq 1$ . Using this fact and integrating by parts  $m$  times, we get

$$|K_1^\pm(\sigma, t)| \leq C_m \sigma^{-1/2} |t \pm \sigma|^{-m}. \quad (2.29)$$

By (2.29), we obtain

$$\begin{aligned} \int_{-\infty}^{\infty} \langle t \rangle^s |K_1^\pm(\sigma, t)| dt &\leq 2\sigma^s \int_{-\infty}^{\infty} |K_1^\pm(\sigma, t)| dt + \int_{-\infty}^{\infty} |t \pm \sigma|^s |K_1^\pm(\sigma, t)| dt \\ &\leq C'_m \sigma^{s-1/2} \int_{-\infty}^{\infty} |t \pm \sigma|^{-m} dt + C_m \sigma^{-1/2} \int_{-\infty}^{\infty} |t \pm \sigma|^{s-m} dt \leq C \sigma^{s-1/2}, \end{aligned}$$

which clearly implies (2.27) in this case.  $\square$

To prove (2.19) we will use the formula

$$e^{it\sqrt{G}} \psi(h^2 G) = (i\pi h)^{-1} \int_0^\infty e^{it\lambda} \varphi_h(\lambda) (R^+(\lambda) - R^-(\lambda)) d\lambda, \quad (2.30)$$

where  $\varphi_h(\lambda) = \varphi_1(h\lambda)$ ,  $\varphi_1(\lambda) = \lambda\psi(\lambda^2)$ . Combining (2.30) together with (2.7), we get

$$V e^{it\sqrt{G}} \psi(h^2 G) = (i\pi h)^{-1} \sum_{\pm} \pm \int_{-\infty}^{\infty} V P_h^\pm(t - \tau) U_h^\pm(\tau) d\tau, \quad (2.31)$$

where

$$\begin{aligned} P_h^\pm(t) &= \int_0^\infty e^{it\lambda} \tilde{\varphi}_h(\lambda) R_0^\pm(\lambda) d\lambda, \\ U_h^\pm(t) &= \int_0^\infty e^{it\lambda} \varphi_h(\lambda) (1 + V R_0^\pm(\lambda))^{-1} d\lambda, \end{aligned}$$

where  $\tilde{\varphi}_h(\lambda) = \tilde{\varphi}_1(h\lambda)$ ,  $\tilde{\varphi}_1 \in C_0^\infty((0, +\infty))$  is such that  $\tilde{\varphi}_1 = 1$  on  $\text{supp } \varphi_1$ . The kernel of the operator  $P_h^\pm(t)$  is of the form  $A_h^\pm(|x - y|, t)$ , where

$$A_h^\pm(\sigma, t) = \pm i 4^{-1} \int_0^\infty e^{it\lambda} \tilde{\varphi}_h(\lambda) H_0^\pm(\sigma\lambda) d\lambda = h^{-1} A_1^\pm(\sigma/h, t/h). \quad (2.32)$$

In the same way as in the proof of Lemma 2.5 one can see that the function  $A_h^\pm$  satisfies the estimate

$$\int_{-\infty}^{\infty} |t|^s |A_h^\pm(\sigma, t)| dt \leq Ch^{1/2} \sigma^{s-1/2} (1 + h^{\epsilon_s} \sigma^{-\epsilon_s}), \quad 0 \leq s \leq 1/2, \quad 0 < h \leq 1, \quad (2.33)$$

where  $\epsilon_s = 0$  if  $0 \leq s < 1/2$ ,  $\epsilon_s = \epsilon$  if  $s = 1/2$ .

Clearly, it suffices to prove (2.19) with  $s = 0$  and  $s = 1/2$ . For these values of  $s$ , using (1.1), (2.31) and (2.33), we obtain

$$\begin{aligned}
& \int_{\mathbf{R}^2} \int_{\mathbf{R}^2} \int_{-\infty}^{\infty} |t|^s \langle |x - y|/h \rangle^{-s} \left| V e^{it\sqrt{G}} \psi(h^2 G) f(x) \right| |g(y)| dt dx dy \\
& \leq Ch^{-1} \sum_{\pm} \int_{\mathbf{R}^2} \int_{\mathbf{R}^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \langle |x - y|/h \rangle^{-s} (|t - \tau|^s + |\tau|^s) \\
& \quad \times |V P_h^{\pm}(t - \tau) U_h^{\pm}(\tau) f(x)| |g(y)| d\tau dt dx dy \\
& \leq Ch^{-1} \sum_{\pm} \int_{\mathbf{R}^2} \int_{\mathbf{R}^2} \int_{\mathbf{R}^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |V(x)| \langle |x - y|/h \rangle^{-s} (|t - \tau|^s + |\tau|^s) \\
& \quad \times |A_h^{\pm}(|x - x'|, t - \tau)| |U_h^{\pm}(\tau) f(x')| |g(y)| d\tau dt dx' dx dy \\
& \leq Ch^{-1} \sum_{\pm} \int_{\mathbf{R}^2} \int_{\mathbf{R}^2} \int_{\mathbf{R}^2} |V(x)| \langle |x - y|/h \rangle^{-s} |g(y)| \\
& \quad \times \left( \int_{-\infty}^{\infty} |\tau|^s |A_h^{\pm}(|x - x'|, \tau)| d\tau \right) \left( \int_{-\infty}^{\infty} |U_h^{\pm}(\tau) f(x')| d\tau \right) dx' dx dy \\
& \quad + Ch^{-1} \sum_{\pm} \int_{\mathbf{R}^2} \int_{\mathbf{R}^2} \int_{\mathbf{R}^2} |V(x)| \langle |x - y|/h \rangle^{-s} |g(y)| \\
& \quad \times \left( \int_{-\infty}^{\infty} |A_h^{\pm}(|x - x'|, \tau)| d\tau \right) \left( \int_{-\infty}^{\infty} |\tau|^s |U_h^{\pm}(\tau) f(x')| d\tau \right) dx' dx dy \\
& \leq Ch^{-1/2} \sum_{\pm} \int_{\mathbf{R}^2} \int_{\mathbf{R}^2} \int_{\mathbf{R}^2} |V(x)| \langle |x - y|/h \rangle^{-s} |x - x'|^{s-1/2} (1 + h^{\epsilon s} |x - x'|^{-\epsilon s}) |g(y)| \\
& \quad \times \left( \int_{-\infty}^{\infty} |U_h^{\pm}(\tau) f(x')| d\tau \right) dx' dx dy \\
& \quad + Ch^{-1/2} \sum_{\pm} \int_{\mathbf{R}^2} \int_{\mathbf{R}^2} \int_{\mathbf{R}^2} |V(x)| \langle |x - y|/h \rangle^{-s} |x - x'|^{-1/2} |g(y)| \\
& \quad \times \left( \int_{-\infty}^{\infty} |\tau|^s |U_h^{\pm}(\tau) f(x')| d\tau \right) dx' dx dy := I_1 + I_2. \tag{2.34}
\end{aligned}$$

To estimate  $I_1$  when  $s = 1/2$ , set  $q = (2\epsilon)^{-1}$ ,  $1/p + 1/q = 1$ , and observe that in view of (1.1) we have the bound

$$\begin{aligned}
& \int_{\mathbf{R}^2} |V(x)| \langle |x - y|/h \rangle^{-1/2} |x - x'|^{-\epsilon} dx \\
& \leq \left( \int_{\mathbf{R}^2} |V(x)| \langle |x - y|/h \rangle^{-p/2} dx \right)^{1/p} \left( \int_{\mathbf{R}^2} |V(x)| |x - x'|^{-1/2} dx \right)^{1/q} \\
& \leq C_1 \left( \int_{\mathbf{R}^2} |V(x)| \langle |x - y|/h \rangle^{-1/2} dx \right)^{1/p} \\
& \leq C_1 h^{1/(2p)} \left( \int_{\mathbf{R}^2} |V(x)| |x - y|^{-1/2} dx \right)^{1/p} \leq C_2 h^{1/2-\epsilon}.
\end{aligned}$$



Thus, we obtain

$$I_1 \leq C' h^{s-1/2} \sum_{\pm} \int_{\mathbf{R}^2} \int_{\mathbf{R}^2} \int_{-\infty}^{\infty} |U_h^{\pm}(\tau) f(x')| |g(y)| d\tau dx' dy. \quad (2.35)$$

To estimate  $I_2$  when  $s = 1/2$ , we use the inequality

$$\langle |x - y|/h \rangle^{-1/2} |x - x'|^{-1/2} \leq \langle |x' - y|/h \rangle^{-1/2} \left( |x - y|^{-1/2} + |x - x'|^{-1/2} \right).$$

We get

$$I_2 \leq C'' h^{-1/2} \sum_{\pm} \int_{\mathbf{R}^2} \int_{\mathbf{R}^2} \int_{-\infty}^{\infty} |\tau|^s \langle |x' - y|/h \rangle^{-s} |U_h^{\pm}(\tau) f(x')| |g(y)| d\tau dx' dy. \quad (2.36)$$

On the other hand, by the identity

$$(1 + V R_0^{\pm}(\lambda))^{-1} = 1 - V R_0^{\pm}(\lambda) (1 + V R_0^{\pm}(\lambda))^{-1},$$

we obtain

$$U_h^{\pm}(t) = \widehat{\varphi}_h(t) - \int_{-\infty}^{\infty} V P_h^{\pm}(t - \tau) U_h^{\pm}(\tau) d\tau. \quad (2.37)$$

Since

$$\widehat{\varphi}_h(t) = h^{-1} \widehat{\varphi}_1(t/h),$$

we have

$$\int_{-\infty}^{\infty} |t|^s |\widehat{\varphi}_h(t)| dt \leq C h^s. \quad (2.38)$$

Using (2.37) and (2.38), in the same way as in the proof of (2.34)-(2.36), we obtain with  $s = 0$  or  $s = 1/2$ ,

$$\begin{aligned} & \int_{\mathbf{R}^2} \int_{\mathbf{R}^2} \int_{-\infty}^{\infty} |t|^s \langle |x - y|/h \rangle^{-s} |U_h^{\pm}(t) f(x)| |g(y)| dt dx dy \leq C h^s \|f\|_{L^1} \|g\|_{L^1} \\ & + C h^{s+1/2} \int_{\mathbf{R}^2} \int_{\mathbf{R}^2} \int_{-\infty}^{\infty} |U_h^{\pm}(\tau) f(x')| |g(y)| d\tau dx' dy \\ & + C h^{1/2} \int_{\mathbf{R}^2} \int_{\mathbf{R}^2} \int_{-\infty}^{\infty} |\tau|^s \langle |x' - y|/h \rangle^{-s} |U_h^{\pm}(\tau) f(x')| |g(y)| d\tau dx' dy. \end{aligned} \quad (2.39)$$

Taking  $h$  small enough we can absorb the second and the third terms in the RHS of (2.39) and get the estimate

$$\int_{\mathbf{R}^2} \int_{\mathbf{R}^2} \int_{-\infty}^{\infty} |t|^s \langle |x - y|/h \rangle^{-s} |U_h^{\pm}(t) f(x)| |g(y)| dt dx dy \leq C' h^s \|f\|_{L^1} \|g\|_{L^1}. \quad (2.40)$$

Now (2.19) follows from (2.34)-(2.36) and (2.40).  $\square$

### 3 Proof of Theorem 1.2

Set

$$\Psi(t; h) = e^{itG} \psi(h^2 G) - e^{itG_0} \psi(h^2 G_0).$$

As in the previous section, one can derive (1.6) from the following

**Proposition 3.1** *Let  $V$  satisfy (1.1). Then, there exist positive constants  $C$  and  $h_0$  so that for  $0 < h \leq h_0$ , we have*

$$\|\Psi(t; h)\|_{L^1 \rightarrow L^\infty} \leq Ch^{1/2}|t|^{-1}, \quad t \neq 0. \quad (3.1)$$

*Proof.* We will derive (3.1) from (2.19). To this end, we will use the identity

$$e^{it\lambda^2} \varphi(h^2\lambda^2) = \int_{-\infty}^{\infty} e^{i\tau\lambda} \zeta_h(t, \tau) d\tau, \quad (3.2)$$

where  $\varphi \in C_0^\infty((0, +\infty))$ ,  $\varphi = 1$  on  $\text{supp } \psi_1$ , the functions  $\psi$  and  $\psi_1$  being as in the previous section, and

$$\zeta_h(t, \tau) = (2\pi)^{-1} \int_0^\infty e^{it\lambda^2 - i\tau\lambda} \varphi(h^2\lambda^2) d\lambda = h^{-1} \zeta_1(th^{-2}, \tau h^{-1}). \quad (3.3)$$

We deduce from (3.2) the formula

$$e^{itG} \psi(h^2G) = \int_{-\infty}^{\infty} \zeta_h(t, \tau) e^{i\tau\sqrt{G}} \psi(h^2G) d\tau. \quad (3.4)$$

Given any integer  $m \geq 0$ , integrating by parts  $m$  times and using the well known bound

$$\left| \int_{-\infty}^{\infty} e^{it\lambda^2 - i\tau\lambda} \phi(\lambda) d\lambda \right| \leq C|t|^{-1/2}, \quad \forall t \neq 0, \tau \in \mathbf{R},$$

where  $\phi \in C_0^\infty(\mathbf{R})$ , one easily obtains the bound

$$|\zeta_1(t, \tau)| \leq C_m |t|^{-m-1/2} \langle \tau \rangle^m, \quad \forall t \neq 0, \tau \in \mathbf{R}. \quad (3.5)$$

By (3.3) and (3.5),

$$|\zeta_h(t, \tau)| \leq C_m h^{2m} |t|^{-m-1/2} \langle \tau/h \rangle^m, \quad \forall t \neq 0, \tau \in \mathbf{R}, h > 0, \quad (3.6)$$

for every integer  $m \geq 0$ , and hence for all real  $m \geq 0$ . By (2.5), (2.20) and (3.4), we get

$$\begin{aligned} |\langle \Psi(t; h) f, g \rangle| &\leq Ch^{1/2} \|\Psi(t; h) f\|_{L^\infty} \|g\|_{L^1} \\ &+ \int_{-\infty}^{\infty} |\zeta_h(t, \tau)| \left| \left\langle e^{i\tau\sqrt{G_0}} \psi(h^2G_0) f, (\psi_1(h^2G) - \psi_1(h^2G_0)) g \right\rangle \right| d\tau \\ &+ \int_{-\infty}^{\infty} |\zeta_h(t, \tau)| \left| \left\langle e^{i\tau\sqrt{G_0}} \psi_1(h^2G_0) (\psi(h^2G) - \psi(h^2G_0)) f, g \right\rangle \right| d\tau \\ &+ \int_{-\infty}^{\infty} |\zeta_h(t, \tau)| \left| \left\langle \sin(\tau\sqrt{G_0}) \psi_1(h^2G_0) (\psi(h^2G) - \psi(h^2G_0)) f, g \right\rangle \right| d\tau \\ &+ \int_{-\infty}^{\infty} |\zeta_h(t, \tau)| \left| \left\langle \sin(\tau\sqrt{G_0}) \tilde{\psi}_1(h^2G_0) (\tilde{\psi}(h^2G) - \tilde{\psi}(h^2G_0)) f, g \right\rangle \right| d\tau \\ &+ h \int_{-\infty}^{\infty} \int_0^\tau |\zeta_h(t, \tau')| \left| \left\langle V e^{i\tau'\sqrt{G}} \psi(h^2G) f, \sin((\tau - \tau')\sqrt{G_0}) \tilde{\psi}_1(h^2G_0) g \right\rangle \right| d\tau' d\tau. \end{aligned} \quad (3.7)$$

Using (3.6) with  $m = 1/2$  and (2.27) with  $s = 1/2$  together with (2.5), we obtain that the first integral in the RHS of (3.7) is upper bounded by

$$Ch|t|^{-1} \int_{\mathbf{R}^2} \int_{\mathbf{R}^2} \int_{-\infty}^{\infty} \langle \tau/h \rangle^{1/2} |K_h(|x-y|, \tau)| |f(x)| |(\psi_1(h^2G) - \psi_1(h^2G_0)) g(y)| d\tau dx dy$$

$$\leq C|t|^{-1} \int_{\mathbf{R}^2} \int_{\mathbf{R}^2} |f(x)| |(\psi_1(h^2 G) - \psi_1(h^2 G_0)) g(y)| dx dy \leq Ch^{1/2} |t|^{-1} \|f\|_{L^1} \|g\|_{L^1},$$

and similarly for the next three integrals. The last term is upper bounded by

$$\begin{aligned} & Ch^2 |t|^{-1} \int_{\mathbf{R}^2} \int_{\mathbf{R}^2} \int_{-\infty}^{\infty} \int_0^{\tau} \left( |\tau'/h|^{1/2} + \langle (\tau - \tau')/h \rangle^{1/2} \right) \left| \tilde{K}_h(|x-y|, (\tau - \tau')) \right| \\ & \quad \times \left| V e^{i\tau' \sqrt{G}} \psi(h^2 G) f(x) \right| |g(y)| d\tau' d\tau dx dy \\ & \leq Ch^{3/2} |t|^{-1} \int_{\mathbf{R}^2} \int_{\mathbf{R}^2} \int_{-\infty}^{\infty} |\tau|^{1/2} \left| V e^{i\tau \sqrt{G}} \psi(h^2 G) f(x) \right| |g(y)| d\tau \int_{-\infty}^{\infty} \left| \tilde{K}_h(|x-y|, \tau) \right| d\tau dx dy \\ & + Ch^2 |t|^{-1} \int_{\mathbf{R}^2} \int_{\mathbf{R}^2} \int_{-\infty}^{\infty} \left| V e^{i\tau \sqrt{G}} \psi(h^2 G) f(x) \right| |g(y)| d\tau \int_{-\infty}^{\infty} \langle \tau/h \rangle^{1/2} \left| \tilde{K}_h(|x-y|, \tau) \right| d\tau dx dy \\ & \leq Ch^{1/2} |t|^{-1} \int_{\mathbf{R}^2} \int_{\mathbf{R}^2} \int_{-\infty}^{\infty} |\tau|^{1/2} \langle |x-y|/h \rangle^{-1/2} \left| V e^{i\tau \sqrt{G}} \psi(h^2 G) f(x) \right| |g(y)| d\tau dx dy \\ & \quad + Ch |t|^{-1} \int_{\mathbf{R}^2} \int_{\mathbf{R}^2} \int_{-\infty}^{\infty} \left| V e^{i\tau \sqrt{G}} \psi(h^2 G) f(x) \right| |g(y)| d\tau dx dy \\ & \leq Ch^{1/2} |t|^{-1} \|f\|_{L^1} \|g\|_{L^1}, \end{aligned}$$

where  $\tilde{K}_h(|x-y|, t)$  denotes the kernel of the operator  $\sin(t\sqrt{G_0}) \tilde{\psi}_1(h^2 G_0)$ , and we have used (2.19) together with the fact that the function  $\tilde{K}_h(\sigma, t)$  satisfies (2.27). Thus, we obtain

$$|\langle \Psi(t; h) f, g \rangle| \leq Ch^{1/2} \|\Psi(t; h) f\|_{L^\infty} \|g\|_{L^1} + Ch^{1/2} |t|^{-1} \|f\|_{L^1} \|g\|_{L^1},$$

which clearly implies (3.1), provided  $h$  is taken small enough.  $\square$

## References

- [1] M. BEALS, *Optimal  $L^\infty$  decay estimates for solutions to the wave equation with a potential*, Commun. Partial Diff. Equations **19** (1994), 1319-1369.
- [2] F. CARDOSO, C. CUEVAS AND G. VODEV, *Dispersive estimates of solutions to the wave equation with a potential in dimensions two and three*, Serdica Math. J. **31** (2005), 263-278.
- [3] P. D'ANCONA AND V. PIERFELICE, *On the wave equation with a large rough potential*, J. Funct. Analysis **227** (2005), 30-77.
- [4] V. GEORGIEV AND N. VISCIGLIA, *Decay estimates for the wave equation with potential*, Commun. Partial Diff. Equations **28** (2003), 1325-1369.
- [5] M. GOLDBERG, *Dispersive bounds for the three dimensional Schrödinger equation with almost critical potentials*, GAFA **16** (2006), 517-536.
- [6] J.-L. JOURNÉ, A. SOFER AND C. SOGGE, *Decay estimates for Schrödinger operators*, Commun. Pure Appl. Math. **44** (1991), 573-604.
- [7] S. MOULIN, *Low frequency dispersive estimates for the wave equation in higher dimensions*, submitted.
- [8] S. MOULIN AND G. VODEV, *Low frequency dispersive estimates for the Schrödinger group in higher dimensions*, Asymptot. Anal., to appear.
- [9] W. SCHLAG, *Dispersive estimates for Schrödinger operators in two dimensions*, Commun. Math. Phys. **257** (2005), 87-117.

- [10] I. RODNIANSKI AND W. SCHLAG, *Time decay for solutions of Schrödinger equations with rough and time-dependent potentials*, Invent. Math. **155** (2004), 451-513.
- [11] G. VODEV, *Dispersive estimates of solutions to the Schrödinger equation in dimensions  $n \geq 4$* , Asymptot. Anal. **49** (2006), 61-86.
- [12] G. VODEV, *Dispersive estimates of solutions to the wave equation with a potential in dimensions  $n \geq 4$* , Commun. Partial Diff. Equations **31** (2006), 1709-1733.

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